Recitation Notes, Part 2

David Menendez (davemm@cs.rutgers.edu)

Since we got a little rushed towards the end of recitation last night, I thought I'd post complete versions of the proofs I was going through at the end.

1 Induction

Recall the definition of the Fibonacci numbers:

$$F(n) = \begin{cases} 1 & \text{if } n = 1 \vee n = 2 \\ F(n-1) + F(n-2) & \text{if } n > 2 \end{cases} \tag{1}$$

This gives the usual sequence 1, 1, 2, 3, 5, 8, 13, ...

We would like to prove the obvious-seeming claim that $F(n) < 2^n$ for all positive n.

1.1 Proof by strong induction

We want to show $F(n) < 2^n$ for all positive integers n.

Basis step. $F(1) = F(2) = 1$, $2^1 = 2$, and $2^2 = 4$. Therefore, $F(1) < 2^1$ and $F(2) < 2^2$.

Inductive step. Assume that $F(i) < 2^i$ for all positive integers $i \leq k$, where $k \geq 2$ is an arbitrary fixed integer. We need to show that $F(k+1) < 2^{k+1}$.

$$\begin{aligned} F(k+1) &= F(k) + F(k-1) && \text{by definition of } F \\ &< 2^k + 2^{k-1} && \text{by our assumption, } F(k) < 2^k \text{ and } F(k-1) < 2^{k-1} \\ &< 2^k + 2^k = 2^{k+1} \end{aligned}$$

Therefore, $F(k+1) < 2^{k+1}$. This concludes the inductive step.

By strong induction, we have shown that $F(n) < 2^n$ for all positive integers n. ■

1.2 Proof by induction using inductive loading

The inductive step in the previous proof has the form $(P(1) \wedge P(2) \wedge \cdots \wedge P(k)) \rightarrow P(k+1)$, but it only used two of its assumptions: $P(k-1)$ and $P(k)$. This suggests that we could write a proof using (non-strong) mathematical induction by a technique known as *inductive loading*, where we prove a stronger statement in order to provide more assumptions in the inductive step.

For example, if we define $Q(k) \equiv P(k) \wedge P(k+1)$, then showing

$$Q(k) \rightarrow Q(k+1) \equiv (P(k) \wedge P(k+1)) \rightarrow (P(k+1) \wedge P(k+2)$$

is a valid inductive step.

So here's a proof that $F(n) < 2^n$ and $F(n+1) < 2^{n+1}$ for all positive integers n.

Basis step. $F(1) = F(2) = 1$, $2^1 = 2$, and $2^2 = 4$. Therefore, $F(1) < 2^1$ and $F(2) < 2^2$.

Inductive step. Assume that $F(k) < 2^k$ and $F(k+1) < 2^{k+1}$ for an arbitrary fixed integer $k \geq 1$. We need to show that $F(k+1) < 2^{k+1}$ and $F(k+2) < 2^{k+2}$.

- $F(k+1) < 2^{k+1}$ is true by assumption.
- $$\begin{aligned} F(k+2) &= F(k+1) + F(k) && \text{by definition of } F \\ &< 2^{k+1} + 2^k && \text{by our assumptions} \\ &< 2^{k+1} + 2^{k+1} = 2^{k+2} \end{aligned}$$

Therefore, $F(k+1) < 2^{k+1}$ and $F(k+2) < 2^{k+2}$. This concludes the inductive step.

By mathematical induction, we have shown that $F(n) < 2^n$ and $F(n+1) < 2^{n+1}$ for all positive integers n. ∎

1.3 Proof by induction using a lemma

Another way to prove $F(n) < 2^n$ relies on knowing an additional fact about $F(n)$. Specifically, that $F(n) \leq F(n+1)$ for all positive n. Using that additional fact (known as a *lemma*), we can prove our claim using normal induction.

We want to prove $F(n) < 2^n$ for all positive integers n.

Case 1 $n = 1$. $F(1) = 1$ and $2^1 = 2$, therefore $F(1) < 2^1$.

Case 2 $n \geq 2$. We will prove this case by induction.

> *Basis step.* $F(2) = 1$ and $2^2 = 4$. Therefore, $F(2) < 2^2$.
>
> *Inductive step.* Assume $F(k) < 2^k$ for some arbitrary fixed integer $k \geq 2$. We need to show $F(k+1) < 2^{k+1}$.

$$
\begin{aligned}
F(k+1) \quad &= F(k) + F(k-1) && \text{by definition of } F \\
&\leq F(k) + F(k) && \text{because } F(k-1) \leq F(k) \\
&< 2^k + 2^k = 2^{k+1} && \text{by our assumption}
\end{aligned}
$$

Therefore, $F(k+1) < 2^{k+1}$. This concludes the inductive step.

By mathematical induction, we have shown that $F(n) < 2^n$ for all integers $n \geq 2$.

Combining both cases, we have shown that $F(n) < 2^n$ for all positive integers. ■

Note that we had to restrict induction to the case where $n \geq 2$. This was because our lemma states $F(n) < F(n+1)$ for positive n and we use it to show $F(k-1) < F(k)$, so we needed to guarantee that $k - 1 \geq 1$.

Proof of the lemma $F(n) \leq F(n+1)$ is left as an exercise.

2 Natural deduction using quantifiers

I was asked for some examples involving quantifiers in natural deduction, specifically about when it is permissible to use universal generalization. Let's compare two proofs, one valid and one invalid.

1	$\exists x \forall y P(x,y)$		1	$\forall x \exists y P(x,y)$	
2	$\forall y P(a,y)$	H	2	$\exists y P(x,y)$	$\forall$E 1,y
3	$P(a,y)$	$\forall$E 2,y	3	$P(x,a)$	H
4	$\exists x P(x,y)$	$\exists$I 3,a	4	$\forall x P(x,a)$	$\forall$I 3
5	$\exists x P(x,y)$	$\exists$E 1,2,4,a	5	$\exists y \forall x P(x,y)$	$\exists$I 4,a
6	$\forall y \exists x P(x,y)$	$\forall$I 5	6	$\exists y \forall x P(x,y)$	$\exists$E 2,3,5,a

Note that I'm writing $P(x,y)$ to match the style of the textbook, rather than Pxy to match the style of the natural deduction guide.

The proof on the left is valid, but the proof on the right contains an error on line 4. The rule of universal generalization ($\forall$I) can't be used there because the variable begin generalized (x) appears in an attainable hypothesis (3). That means we're assuming something about x at that point (specifically, that $P(x,a)$ is true), so it isn't sufficiently arbitrary to generalize.

Here are some more involved examples:

1	$\neg\exists x P(x)$	
2	$\mid\ P(x)$	H
3	$\mid\ \exists x P(x)$	$\exists$I 2,x
4	$\mid\ \neg\exists x P(x)$	IT 1
5	$\neg P(x)$	$\neg$I 2,3,4
6	$\forall x \neg P(x)$	$\forall$I 5

1	$\neg\forall x P(x)$	
2	$\mid\ \neg\exists x \neg P(x)$	H
3	$\mid\ \mid\ \neg P(x)$	H
4	$\mid\ \mid\ \exists x \neg P(x)$	$\exists$I 3,x
5	$\mid\ \mid\ \neg\exists x \neg P(x)$	IT 2
6	$\mid\ \neg\neg P(x)$	$\neg$I 3,4,5
7	$\mid\ P(x)$	$\neg$E 6
8	$\mid\ \forall x P(x)$	$\forall$I 7
9	$\mid\ \neg\forall x P(x)$	IT 1
10	$\neg\neg\exists x \neg P(x)$	$\neg$I 2,8,9
11	$\exists x \neg P(x)$	$\neg$E 10

1	$\forall x \neg P(x)$	
2	$\mid\ \exists x P(x)$	H
3	$\mid\ \mid\ P(a)$	H
4	$\mid\ \mid\ \mid\ \neg\Box$	H
5	$\mid\ \mid\ \mid\ P(a)$	IT 3
6	$\mid\ \mid\ \mid\ \neg P(a)$	$\forall$E 1,a
7	$\mid\ \mid\ \neg\neg\Box$	$\neg$I 4,5,6
8	$\mid\ \mid\ \Box$	$\neg$E 7
9	$\mid\ \Box$	$\exists$E 2,3,8,a
10	$\mid\ Q$	$\Box$E 9
11	$\mid\ \neg Q$	$\Box$E 9
12	$\neg\exists x P(x)$	$\neg$I 2,10,11

1	$\exists x \neg P(x)$	
2	$\mid\ \forall x P(x)$	H
3	$\mid\ \mid\ \neg P(a)$	H
4	$\mid\ \mid\ \mid\ \neg\Box$	H
5	$\mid\ \mid\ \mid\ P(a)$	$\forall$E 2
6	$\mid\ \mid\ \mid\ \neg P(a)$	IT 3
7	$\mid\ \mid\ \neg\neg\Box$	$\neg$I 4,5,6
8	$\mid\ \mid\ \Box$	$\neg$E 7
9	$\mid\ \Box$	$\exists$E 1,3,8,a
10	$\mid\ Q$	$\Box$E 9
11	$\mid\ \neg Q$	$\Box$E 9
12	$\neg\forall x P(x)$	$\neg$I 2,10,11

Note the explicit introduction of false ($\Box$), in order to pass the contradiction through existential elimination ($\exists$E). It's possible there is a more elegant way to do that.