Recitation sample problems

David Menendez (davemm@cs.rutgers.edu)

Some worked examples I wrote up for past recitations.

1 Mathematical Induction

Unfortunately, all the examples I've done in class are from the book, and we're not supposed to post answers on-line. So here's a proof by induction of a well-known theorem:

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

Proof: By induction.

Basis step: We need to show that $1 = \frac{1(1+1)}{2}$. $\frac{1(1+1)}{2} = \frac{2}{2} = 1$.

Inductive step: Assume that $1 + 2 + \cdots + k = \frac{k(k+1)}{2}$ for an arbitrary integer $k \geq 1$.[1] We need to show that $1 + 2 + \cdots + (k+1) = \frac{(k+1)(k+2)}{2}$.

$$\begin{aligned} 1 + 2 + \cdots + k + (k+1) &= \frac{k(k+1)}{2} + (k+1) && \text{by our assumption} \\ &= \frac{k(k+1) + 2(k+1)}{2} \\ &= \frac{(k+1)(k+2)}{2} \end{aligned}$$

This completes the inductive step.

Thus, we conclude that $1 + \cdots + n = \frac{n(n+1)}{2}$ for all integers $n \geq 1$.

1.1 Strong induction

Let's say we need to write a simple program to raise an integer to a power. That is, for some m and n, we need to calculate m^n. The easiest way to do this is to multiply m by itself $n - 1$ times,

[1] This is the inductive hypothesis.

so that $m^4 = m \cdot m \cdot m \cdot m$. I claim we can use fewer multiplications by setting $m^2 = m \cdot m$ and $m^4 = m^2 \cdot m^2$ and so forth.

In general,

$$m^n = \begin{cases} m^k \cdot m^k & \text{if } n = 2k \\ m^k \cdot m^k \cdot m & \text{if } n = 2k+1 \end{cases}$$

So calculating m^n requires calculating $m^{\lfloor n/2 \rfloor}$ and then performing 1 or 2 additional multiplications.

Let's use strong induction to prove that this method requires fewer than $n-1$ multiplications to calculate m^n for $n \geq 4$.

Hypothesis[2]*:* For all integers $n \geq 4$, m^n can be calculated with fewer than $n-1$ multiplications.

Basis step: To find m^4, we first calculate $m^2 = m \cdot m$ and then $m^4 = m^2 \cdot m^2$ This requires 2 multiplications, which is fewer than 4 - 1 = 3.

Inductive step: Let k be an arbitrary integer greater than or equal to 4. Assume that we can calculate m^i in fewer than i multiplications for all i such that $4 \leq i \leq k$. We need to show that we can calculate m^{k+1} with fewer than k multiplications. Because our method involves calculating $m^{\lfloor (k+1)/2 \rfloor}$, we will need to consider whether $k+1$ is odd or even, and separately consider the cases where $\lfloor (k+1)/2 \rfloor < 4$:

- *Case 1:* $k = 4$. We will calculate $m^5 = m^2 \cdot m^2 \cdot m$. Since finding m^2 requires 1 multiplication, we will need 3 to calculate m^5.
- *Case 2:* $k = 5$. We will calculate $m^6 = m^3 \cdot m^3$. Since finding m^3 requires 2 multiplications, we will need 3 to calculate m^6.
- *Case 3:* $k = 6$. We will calculate $m^7 = m^3 \cdot m^3 \cdot m$. Since finding m^3 requires 2 multiplications, we will need 4 to calculate m^7.
- *Case 4:* $k \geq 7$. In this case, $\lfloor (k+1)/2 \rfloor \geq 4$, so we may use the inductive hypothesis. We will consider two sub-cases, where $k+1$ is even or odd:
  - *Case 4.1:* $k+1$ is even. Since $k+1$ is even, there must be an integer j such that $k+1 = 2j$. Therefore, we will calculate $m^{k+1} = m^j \cdot m^j$. By the inductive hypothesis, we can calculate m^j in fewer than $j-1$ multiplications. Therefore, we need fewer than j multiplications to calculate m^{k+1}.

    Since $k \geq 7$, $j < k$ and therefore m^{k+1} can be calculated in fewer than k multiplications.
  - *Case 4.2:* $k+1$ is odd. Since $k+1$ is odd, there must be an integer j such that $k+1 = 2j+1$. Therefore, we will calculate $m^{k+1} = m^j \cdot m^j \cdot m$. By the inductive hypothesis, we can calculate m^j in fewer than $j-1$ multiplications. Therefore, we need fewer than $j+1$ multiplications to calculate m^{k+1}.

[2] This is not the inductive hypothesis!

Since $k \geq 7$, $j + 1 < k$, and therefore m^{k+1} can be calculated in fewer than k multiplications.

All the cases show that m^{k+1} can be calculated in fewer than k multiplications. This completes the inductive step.

Thus, by strong induction, m^n can be calculated with fewer than $n - 1$ multiplications for all integers $n \geq 4$.

2 Natural Deduction

2.1 One problem three ways

Normally, if you needed to show something like $\neg(P \vee Q) \rightarrow (\neg P \wedge \neg Q)$, you would simply note that this is an obvious consequence of DeMorgan's laws and that would be sufficient. But since we're learning how to prove things, here are two less-implicit proofs.

First, using logical equivalences:

$$\begin{aligned}
\neg(P \vee Q) \rightarrow (\neg P \wedge \neg Q) &\equiv (\neg P \wedge \neg Q) \rightarrow (\neg P \wedge \neg Q) && \text{DeMorgan's Laws} \\
&\equiv \neg(\neg P \wedge \neg Q) \vee (\neg P \wedge \neg Q) && \rightarrow \text{Equivalence} \\
&\equiv \text{True} && \text{Negation}
\end{aligned}$$

Here, we used DeMorgan's laws to replace $\neg(P \vee Q)$ with $(\neg P \wedge \neg Q)$, then we rewrote the implication into a disjunction, which was trivially true because one term was the negation of the other.

Next, using natural deduction.

1	$\neg(P \vee Q)$	
2	P	
3	$P \vee Q$	$\vee$I 2
4	False	False Intro 1,3
5	$\neg P$	$\neg$I 2,4
6	Q	
7	$P \vee Q$	$\vee$I 6
8	False	False Intro 1,7
9	$\neg Q$	$\neg$I 6,8
10	$\neg P \wedge \neg Q$	$\wedge$I 5,9
11	$\neg(P \vee Q) \rightarrow (\neg P \wedge \neg Q)$	$\rightarrow$I 1,10

We want to show an implication, so we assume the premise (1) and attempt to show the consequence (10). Since the consequence is a conjunction, we need to show both parts (5,9). These are negations, so we can prove them by assuming the opposite (2,6) and showing a contradiction (3 and 7 contradict 1).

2.2 Some more examples

$(\neg P \vee \neg Q) \to \neg(P \vee Q)$:

1	$\neg P \vee \neg Q$	
2	$P \wedge Q$	
3	$\neg P$	
4	P	$\wedge$E 2
5	False	False Intro 3,4
6	$\neg Q$	
7	Q	$\wedge$E 2
8	False	False Intro 6,7
9	False	$\vee$E 1,5,8
10	$\neg(P \wedge Q)$	$\neg$I 2,9
11	$(\neg P \vee \neg Q) \to \neg(P \wedge Q)$	$\to$I 1,10

Here, since we are trying to conclude something from a disjunction, we have to use or-elimination by showing that we can reach the same conclusion (in this case a contradiction) from both terms.

$((P \vee Q) \wedge R) \rightarrow ((P \wedge R) \vee (Q \wedge R))$:

1	$(P \vee Q) \wedge R$	
2	R	$\wedge$E 1
3	$P \vee Q$	$\wedge$E 1
4	P	
5	$P \wedge R$	$\wedge$I 2,4
6	$(P \wedge R) \vee (P \wedge Q)$	$\vee$I 5
7	Q	
8	$Q \wedge R$	$\wedge$I 2,7
9	$(P \wedge R) \vee (Q \wedge R)$	$\vee$I 8
10	$(P \wedge R) \vee (Q \wedge R)$	$\vee$E 3,6,9
11	$((P \vee Q) \wedge R) \rightarrow ((P \wedge R) \vee (Q \wedge R))$	$\rightarrow$I 1,11

$((P \wedge Q) \vee R) \rightarrow ((P \vee R) \wedge (Q \vee R))$:

1	$(P \wedge Q) \vee R$	
2	$P \wedge Q$	
3	P	$\wedge$E 2
4	$P \vee R$	$\vee$I 3
5	Q	$\wedge$E 2
6	$Q \vee R$	$\vee$I 5
7	$(P \vee R) \wedge (Q \vee R)$	$\vee$I 4,6
8	R	
9	$P \vee R$	$\vee$I 8
10	$Q \vee R$	$\vee$I 8
11	$(P \vee R) \wedge (Q \vee R)$	$\wedge$I 9,10
12	$(P \vee R)(Q \vee R)$	$\vee$E 1,7,11
13	$((P \wedge Q) \vee R) \rightarrow ((P \vee R) \wedge (Q \vee R))$	$\rightarrow$1,12

$P \to (Q \vee R), (P \wedge Q) \to R \vdash P \to R$:

1	$P \to (Q \vee R)$	Premise
2	$(P \wedge Q) \to R$	Premise
3	$\quad \mid\ \underline{P}$	
4	$\quad \mid\ Q \vee R$	$\to$E 1,3
5	$\quad \mid\ \mid\ \underline{Q}$	
6	$\quad \mid\ \mid\ P \wedge Q$	$\wedge$I 3,5
7	$\quad \mid\ \mid\ R$	$\to$E 2,6
8	$\quad \mid\ \mid\ \underline{R}$	
9	$\quad \mid\ \mid\ R$	It 8
10	$\quad \mid\ R$	$\vee$E 4,7,9
11	$P \to R$	$\to$I 3,10