Answers for Quiz 4

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1. 1. $(p \to q) \to r \vdash (q \to (p \to r))$

1	$(p \to q) \to r$	
2	q	H
3	p	H
4	$q \to r$	$\to$E 1,3
5	r	$\to$E 2,4
6	$p \to r$	$\to$I 3,5
7	$q \to (p \to r)$	$\to$I 2,6

   2. $(p \to q) \vdash ((q \to r) \to (p \to r))$

1	$p \to q$	
2	$q \to r$	H
3	p	H
4	q	$\to$E 1,3
5	r	$\to$E 2,4
6	$p \to r$	$\to$I 3,5
7	$(q \to r) \to (p \to r)$	$\to$I 2,6

2.
   - In a proof by *contradiction*, we show that the negation of the hypothesis leads to a contradiction. To prove $p \to q$, we would show $p \wedge \neg q \to$ False.
   - In a proof by *contraposition*, we would show that the negation of the conclusion leads to the negation of the premise. To prove $p \to q$, we would show $\neg q \to \neg p$.

3. (a) Given an integer n, we want to show that $3n^2 + n + 14$ is even.

   Proof by cases:

   - Case 1: n is even. For n to be even, there must exist an integer k such that $n = 2k$. By substitution:

$$
\begin{aligned}
3n^2 + n + 14 &= 3(2k)^2 + (2k) + 14 \\
&= 12k^2 + 2k + 14 \\
&= 2(6k^2 + k + 7)
\end{aligned}
$$

We have shown $3n^2 + n + 14$ is twice some other integer, therefore it must be even.

- Case 2: n is odd. For n to be odd, there must exist an integer k such that $n = 2k + 1$. By substitution:

$$
\begin{aligned}
3n^2 + n + 14 &= 3(2k + 1)^2 + (2k + 1) + 14 \\
&= 3(4k^2 + 4k + 1) + 2k + 15 \\
&= 12k^2 + 12k + 3 + 2k + 15 \\
&= 12k^2 + 14k + 18 \\
&= 2(6k^2 + 7k + 9)
\end{aligned}
$$

We have shown $3n^2 + n + 14$ is twice some other integer, therefore it must be even.

We have shown that $3n^2 + n + 14$ must be even if n is even or odd. Since every integer must be even or odd, we have shown that $3n^2 + n + 14$ is even for any integer.

(b) We want to show that $x^2 - y^2 = 10$ has no positive integer solutions.

Proof by contradiction: Assume there are positive integers x and y such that $x^2 - y^2 = 10$. We can show that $x^2 - y^2 = (x + y)(x - y)$. Let $a = x + y$ and $b = x - y$. Because x and y are positive integers, a must also be a positive integer. Similarly, b must be an integer, and b must be positive because $ab = 10$.

There are only four positive integers which divide evenly into 10: 1, 2, 5, and 10. We will proceed by cases:

- Case 1: $a = 1$ and $b = 10$. We get the equations,

$$
\begin{aligned}
x + y &= 1 \\
x - y &= 10
\end{aligned}
$$

Subtracting them, we get $2y = -9$ and therefore $y = -9/2$. This contradicts the assumption that y is a positive integer.

- Case 2: $a = 2$ and $b = 5$. Subtracting the equations gives $2y = -3$, and therefore $y = -3/2$. This contradicts the assumption that y is a positive integer.
- Case 3: $a = 5$ and $b = 2$. Subtracting the equations gives $2y = 3$, and therefore $y = 3/2$. This contradicts the assumption that y is a positive integer.
- Case 4: $a = 10$ and $b = 1$. Subtracting the equations gives $2y = 9$, and therefore $y = 9/2$. This contradicts the assumption that y is a positive integer.

Since every case leads to a contradiction, we conclude that our hypothesis that x and y are positive integers such that $x^2 - y^2 = 10$ is false. Therefore, no such positive integers exist.