Midterm

CS-205 Section-01
7/19/2012 **Full name (with middle intial):** $\underbrace{\hspace{6cm}}_{\text{by writing my name i swear by the honor code}}$

Read all of the following information before starting the exam:

- This is a closed book exam. For inference rules and logic you could use the cheat sheat provided to you in the class.
- Show all work, clearly and in order for full credit. There are partial points for each steps involved.
- Justify your answers algebraically whenever possible to ensure full credit. When you do use your calculator, sketch all relevant graphs and explain all relevant mathematics.
- Circle or otherwise indicate your final answers.
- Please keep your written answers brief; be clear and to the point.
- There is enough space provided at the bottom of each question for your answers. If this space is not enough, you can use the back side of the page but please do indicate this on the front page near the problem.
- There is a scrap page at the end for calculations and rough writing. (This page will not be graded.)
- This test has 9 problems and is worth 100 points, plus some extra credit (10 points) at the end. It is your responsibility to make sure that you have all of the pages!
- Good luck!

1. (*11 points*)

a. (*5 pts*) Let $A = \{1,3,5,7,9\}$, $B = \{4,5,7,9,10\}$, $C = \{2,4,6,8,10\}$, $D = \{1,2,3\}$ and let the universal set $U = \{1,2,3,4,5,6,7,8,9,10\}$.

(a) $|A| = 5$

(b) $A \cap B = \{5,7,9\}$

(c) $\overline{B} = \{1,2,3,6,8\}$

(d) $(A \cup B) - C = \{1,3,5,7,9\}$

(e) $A - (B \cup C) = \{1,3\}$

(f) $(B \cap \overline{A}) \cup (B \cap \overline{C}) = \{2,5,7,9,10\}$

(g) $\overline{A \cup C} = \{\}$

(h) $\overline{A \cap C} = \{1,2,3,4,5,6,7,8,9,10\}$

(i) $P(D) = \{\{\},\{1\},\{2\},\{3\},\{1,2\},\{1,3\},\{2,3\},\{1,2,3\}\}$

(j) $D \times (B \cap C) = \{(1,4),(1,10),(2,4),(2,10),(3,4),(3,10)\}$

b. (*6 pts*) Prove: if $A \subseteq \overline{B}$, then $B \subseteq \overline{A}$.

For an element x, we have:

$$\begin{aligned} A \subseteq \overline{B} &\equiv (x \in A) \to (x \in \overline{B}) \\ &\equiv (x \in A) \to \neg(x \in B) \\ &\equiv (x \in B) \to \neg(x \in A) \\ &\equiv (x \in B) \to (x \in \overline{A}) \\ &\equiv B \subseteq \overline{A} \end{aligned}$$

Since $A \subseteq \overline{B}$ and $B \subseteq \overline{A}$ are logically equivalent, it follows that if one is true, then the other must be true.

2. (*11 points*)

a. (*5 pts*) Let p, q and r be the propositions such that,

p: "You get an A on the final exam"
q: "You do every exercise in this book"
r: "You get an A in this class"

Write the following propositions using p, q, r and logical connectives.

(a) If you do every exercise in this book then you will get an A in this class.
$q \to r$

(b) You get an A in class only if you get an A in the final exam.
$r \to p$

(c) To get an A in this class it is necessary for you to get an A in the final.
$p \to r$

(d) Getting an A in the final and doing every exercise in this book is sufficient for getting an A in this class.
$(p \land q) \to r$

(e) You will get an A in this class if and only if you either do every exercise in this book or you get an A on the final exam.
$r \leftrightarrow (p \lor q)$

b. (*6 pts*) The set is propositional formulas can be divided into three exclusive types:

(a) **tautology**: a formula that always evaluates to TRUE

(b) **fallacy**: a formula that always evaluates to FALSE

(c) **contingency**: a formula that sometimes evaluates to TRUE, and other times to FALSE.

For each of the following propositions, indicate whether it is a tautology, fallacy or a contingency.

1. p
   contingency

2. $p \to q$
   contingency

3. $p \lor \neg p$
   tautology

4. $p \land \neg p$
   fallacy

5. $p \to (q \to p)$
   tautology

6. $(p \to q) \to (q \to p)$
   contingency

3. (*11 points*)

a. (*5 pts*) Let the universe of discourse consist of all students in your class.

Let,

x: students
y: courses
$F(x)$: "x is a Freshman"
$C(x)$: "x is a Computer Science major"
$M(y)$: "y is a math course"
$T(x, y)$: "x is taking y

Express the statements below in terms of the given predicates defined above, quantifiers and logical connectives.

(a) Ben is a Computer Science major.
$C(Ben)$

(b) Some Freshman is taking Calculus 3.
$\exists x(F(x) \wedge T(x, Calc3))$

(c) Every Computer Science major is taking at least one math course.
$\forall x(C(x) \rightarrow \exists y(M(y) \wedge T(x, y)))$

(d) Every course has a student in it who is not a Computer Science major.
$\forall y \exists x(T(x, y) \wedge \neg C(x))$

(e) No Freshman is taking every math course.
$\neg \exists x(F(x) \wedge \forall y(T(x, y) \wedge M(y)))$

b. (*6 pts*) (i) Rewrite the following formula so that negations only occur next to a predicate function. In other words, move the negations inward.

$$\begin{aligned} &\neg(\forall x \exists y Q(x, y) \vee \forall x R(x)) \qquad (1)\\ \equiv\ &\exists x \forall y \neg Q(x, y) \wedge \exists x \neg R(x) \end{aligned}$$

(ii) Construct an example which shows that,

$$\exists x \forall y P(x, y) \not\leftrightarrow \forall x \exists y P(x, y) \qquad (2)$$

Assume the definitons as below:

x: students
y: college
$P(x, y)$: "x goes to college y"

Let x range over $\{Alice, Bob\}$ and y over $\{Rutgers, Rowan\}$. Let $P(Alice, Rutgers)$ and $P(Bob, Rowan)$ be true, and $P(Alice, Rowan)$ and $P(Bob, Rutgers)$ be false. Now $\forall x \exists y P(x, y)$ is true, but $\exists y \forall x P(x, y)$ is false.

4. (*11 points*)

a. (*5 pts*) Determine the truth value of each of the following statements if the universe of discourse is the set of real numbers (explain your answers)

(a) $\exists x \exists y((x + 2y = 2) \wedge (2x + 4y = 5))$
False. Note that $(x + 2y = 2) \equiv (2x + 4y = 4)$. If such x, y existed, then by transitivity we would have $4 = 5$.

(b) $\forall x \exists y(y^2 - x = 1)$
False. Choose $x = -2$, then $y^2 = -3$, but $\sqrt{-3} \notin \mathbb{R}$.

(c) $\forall x \forall y \exists z(z = (x + 2)/2)$
True. Addition and division by two are closed over the reals, so a z can be chosen for any x and y

(d) $\exists! x(x^2 = 1)$
False. $1^2 = (-1)^2 = 1$

(e) $\exists! x \neg P(x) \rightarrow \neg \forall x P(x)$
True. If a unique x satisfying $\neg P(x)$ exists, then obviously $\exists x \neg P(x)$, which is equivalent to $\neg \forall x P(x)$ by DeMorgan's laws.

b. (*6 pts*) Using natural deduction rules for quantifiers given the premises, $\exists x(P(x) \wedge \neg Q(x))$ and $\forall x(P(x) \rightarrow R(x))$, conclude $\exists x(R(x) \wedge \neg Q(x))$. Write down your solution in detail, inference rule you use and also the corresponding step numbers.

1	$\exists x(P(x) \wedge \neg Q(x))$	
2	$\forall x(P(x) \rightarrow R(x))$	
3	$P(a) \wedge \neg Q(a)$	H
4	$\neg Q(a)$	$\wedge$E 3
5	$P(a)$	$\wedge$E 3
6	$P(a) \rightarrow R(a)$	$\forall$E 2,a
7	$R(a)$	$\rightarrow$E 5,6
8	$R(a) \wedge \neg Q(a)$	$\wedge$I 4,7
9	$\exists x(R(x) \wedge \neg Q(x))$	$\exists$I 8,a
10	$\exists x(R(x) \wedge \neg Q(x))$	$\exists$E 1,3,9,a

5. (*12 points*)

a. (*6 pts*) Using natural deduction rules for quantifiers given the premises, $p \to q$ and $q \to r$, conclude $p \to (q \wedge r)$. Write down your solution in detail, inference rule you use and also the corresponding step numbers.

1	$p \to q$	
2	$q \to r$	
3	$\quad p$	H
4	$\quad q$	$\to$E 1,3
5	$\quad r$	$\to$E 2,4
6	$\quad q \wedge r$	$\wedge$I 4,5
7	$p \to (q \wedge r)$	$\to$I 3,6

b. (*6 pts*) Determine whether the following argument is valid:

1. I play golf or tennis.

2. If it is not Sunday, I play golf and tennis.

3. If it is Saturday or Sunday, then I don't play golf.

4. Therefore, I don't play golf.

Write down the argument using the following variables:

g: I play golf.
t: I play tennis.
s: It is Saturday.
u: It is Sunday.
and determine if the conclusion in the final step of the argument is valid.

$$\begin{array}{l} g \vee t \\ \neg u \to (g \wedge t) \\ (s \vee u) \to \neg g \\ \hline \neg g \end{array}$$

The truth assignment $g = True$, $t = False$, $s = False$, and $u = False$ satisfies the premises but not the conclusion. Therefore, the argument is invalid.

6. (*11 points*)

a. (*5 pts*) Prove that the square of every even integer ends in 0, 4, or 6. (Hint: Every even integer n can be written as $n = 10k + r$ where $r = 0, 2, 4, 6, 8$).

Let n be an even integer. Therefore, $n = 10k + r$ where $r \in \{0, 2, 4, 6, 8\}$ and k is an integer. Thus, $n^2 = (10k + r)^2 = 100k^2 + 20kr + r = 10(10k^2 + 2kr) + r^2$. The factor of 10 means that the last digit of n^2 will be the last digit of r^2. Because $r^2 \in \{0, 4, 16, 36, 64\}$, n^2 must end in 0, 4, or 6.

b. (*6 pts*) Using the definitions of even and odd integer, give an indirect proof (i.e. proof by contradiction) that the following statement is true for all integers n:

if n is odd, then $5n + 3$ is even.

Assume that n is odd and that $5n + 3$ is odd. Therefore, $n = 2k + 1$, for some integer k Thus,

$$\begin{aligned} 5n + 3 &= 5(2k + 1) + 3 \\ &= 10k + 5 + 3 \\ &= 10k + 8 \\ &= 2(5k + 4) \end{aligned}$$

Thus, $5n + 3$ is even. This contradicts our assumption that $5n + 3$ is odd. Therefore, if n is odd, then $5n + 3$ is even.

7. (*11 points*)

a. (*5 pts*) Find/Define a function $f\colon \mathbf{Z} \to \mathbf{N}$ that is 1-1 but not onto.

$$f(x) = \begin{cases} 2(-x)+1 & \text{if } x < 0 \\ 2x+2 & \text{if } x \geq 0 \end{cases}$$

Note that this maps negative integers to odd naturals and non-negative integers to even naturals, so it is 1-1, and that no integer is mapped to 1, so it is not onto.

x	$f(x)$
⋮	⋮
-3	7
-2	5
-1	3
0	2
1	4
2	6
3	10
⋮	⋮

b. (*6 pts*) Describe a rule for generating the terms of each of these sequences $a_0, a_1, a_2, \ldots$:

(a) $2, 3, 6, 11, 18, 27, 38, 51, 66, 83, 102, 123, \ldots$
$a_i = i^2 + 2$

(b) **(Corrected)** $3, \mathbf{4}, 6, 8, 12, 14, 18, 20, 24, 30, 32, 38, 42, 44, 48, 54, \ldots$
The nth element, a_{n-1}, is one more than the nth prime number, for all integers $n \geq 1$.

8. (*12 points*)

a. (*6 pts*) Use mathematical induction to prove that for all integers $n \geq 0$, $2^{3n} - 1$ is divisible by 7.

Basis Step Show $2^{3 \cdot 0} - 1 = 0$ divisible by 7. Trivial.
Inductive Step Assume for an arbitrary fixed integer $k \geq 0$, that $2^{3k} - 1$ is divisible by 7. Show that $2^{3(k+1)} - 1$ is divisible by 7.
Since $2^{3k} - 1$ is divisible by 7, there must be some integer i such that $2^{3k} - 1 = 7i$.

$$\begin{aligned} 2^{3(k+1)} - 1 &= 2^3 \cdot 2^{3k} - 1 \\ &= 8(7i + 1) - 1 \\ &= 56i + 8 - 1 \\ &= 7(8i + 1) \end{aligned}$$

Thus, $2^{3(k+1)} - 1$ is divisible by 7. This concludes the inductive step.
Therefore, by mathematical induction, $2^{3n} - 1$ is divisible by 7 for all integers $n \geq 0$.

b. (*6 pts*) Use mathematical induction to show that,

$$\sum_{i=1}^{n+1} i \cdot 2^i = n \cdot 2^{n+2} + 2 \tag{3}$$

for all integers $n \geq 0$.
Basis Step Show $\sum_{i=1}^{0+1} i \cdot 2^i = 0 \cdot 2^{0+2} + 2$.
$\sum_{i=1}^{0+1} i \cdot 2^i = 1 \cdot 2^1 = 2 = 0 \cdot 2^{0+2} + 2$
Inductive Step Assume $\sum_{i=1}^{k+1} i \cdot 2^i = k \cdot 2^{k+1} + 2$, for an arbitrary fixed integer $k \geq 0$. Show that $\sum_{i=1}^{k+2} i \cdot 2^i = (k+1) \cdot 2^{k+3} + 2$.

$$\begin{aligned} \sum_{i=1}^{k+2} i \cdot 2^i &= \left(\sum_{i=1}^{k+1} i \cdot 2^i \right) + (k+2)2^{k+2} \\ &= k2^{k+2} + 2 + (k+2)2^{k+2} \\ &= (2k+2)2^{k+2} + 2 \\ &= (k+1)2^{k+3} + 2 \end{aligned}$$

This concludes the inductive step.
Therefore, by mathematical induction, $\sum_{i=1}^{n+1} i \cdot 2^i = n \cdot 2^{n+2} + 2$ for all integers $n \geq 0$.

9. (*10 points*) Let $P(n)$ be the statement that a postage of n cents can be made using just 4-cent and 7-cent stamps. The parts of this exercise outline a strong induction proof that $P(n)$ is true for $n \geq 18$.

1. Show the statements $P(18)$, $P(19)$, $P(20)$, and $P(21)$ are true for all $n \geq 18$.

$$P(18) : 2 \cdot 7 + 4 = 18$$
$$P(19) : 1 \cdot 7 + 3 \cdot 4 = 19$$
$$P(20) : 5 \cdot 4 = 20$$
$$P(21) : 3 \cdot 7 = 21$$

2. Write down the inductive hypothesis of the proof ?.

   For an arbitrary fixed integer $k \geq 21$, postage of i cents can be made using 4- and 7-cent stamps, for all integers i where $18 \leq i \leq k$.

3. What statement do you need to prove in the inductive step ?

   Postage of $k+1$ cents can be made using 4- and 7-cent stamps.

4. Complete the inductive step for $k \geq 21$.

   Postage for $k+1$ cents can be made by adding one 4-cent stamp to the postage for $k-3$ cents.

5. Explain why the inductive steps show that this statement is true whenever $n \geq 18$.

   We have shown $P(18), P(19), P(20)$, and $P(21)$, and that $P(k-3) \rightarrow P(k+1)$ for $k \geq 21$. So $P(18) \rightarrow P(22)$, $P(19) \rightarrow P(23)$, $P(20) \rightarrow P(24)$, $P(21) \rightarrow P(25)$, $P(22) \rightarrow P(26)$, and so forth.

Bonus Question (10 Extra Credit Points):

Prove the following theorem we proved in class using strong induction: A simple polygon with n vertices, where n is an integer with $n \geq 3$, can be triangulated into $n-2$ triangles. (You can use the fact without proving that, every simple polygon has an interior diagonal.)

Basis step A triangle can be triangulated into one triangle. Trivial.
Inductive step Assume, for an arbitrary fixed integer $k \geq 3$, that any simple polygon with i sides can be triangulated into $i-2$ triangles, for any integer i where $3 \leq i \leq k$. We need to show that a simple polygon with $k+1$ sides can be triangulated into $k-1$ triangles.
Using an interior diagonal, we can divide the $(k+1)$-gon into an $(s+1)$-gon and an $(r+1)$-gon, where $s+r=k+1$ (The additional sides are the diagonal.) By our assumption, these can be triangulated into $s-1$ and $r-1$ triangles, respectively. Since the two partitions do not overlap, this is a triangulation of the original $(k+1)$-gon in $(s-1)+(r-1)=s+r-2=k-1$ triangles. This concludes the inductive step.
Thus, by strong induction, we have shown that a simple polygon with n sides can be triangulated into $n-2$ triangles, where $n \geq 3$.

Scrap Page
(please do not remove this page from the test packet)